On the instantaneous life of a nondurable input:
a reflection in the light of Cantor and Zeno

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Abstract

That a nondurable input incurs no interest cost is taken as a natural result of the continuous-time model-setting where the input is used up at an instant. We investigate some logical issues arising from this feature: counting the number of periods of production for an instantaneous nondurable input as unity (turning on set theory), counting that for a (any) durable input as uncountably infinite (*pace* Cantor), and accommodating no sequence of production (*pro* Zeno). We conclude that the current accounting practice based on the instantaneous period of production should be revised.

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Hicks (1973) has argued that, whereas in the setting of discrete time (his "weeks") "interest on the net input of the week (since it was paid for on the Monday) is included in the capital at the end of the week," "[i]f we had worked with continuous time, this interest would have disappeared"; "[b]y shrinking the week, we can cause it to disappear" (p. 30). He also considers the case in which a productive process is "truncated" in the first "week" of operation so that the economically meaningful part of the process is represented, in his neo-Austrian framework, by a singleton of net output; in this case, "input produces output instantaneously" (p. 21) and the discounted capital value of net output is outright equal to the net output itself—the net output is "undiscounted, because it comes in immediately" (p.

20).

This "theoretical" argument of Hicks's must be, if implicitly, the basis of the not-so-theoretical statement of Silberberg and Suen (2001) in their widely-read textbook: "Capital can consist of intermediate goods, which are used only once in some productive process, but interest rates and time preference are not directly relevant for such goods. Most interest (no pun intended) in capital goods focuses on those goods that last over several or many time periods" (p. 387).

The Hicksian argument must also be the ground on which many recent important models use, in the setting of continuous time, the following type of accounting relationship, which refers to a productive process (or an aggregate economy) where *both* a nondurable input and a durable input are utilized:

(1) $$Y = wL + p_n K_n + r p_d K_d ,$$

where Y = output, L = labor, K_n = nondurable input, K_d = (ever-lasting) durable input, w = wage rate, r = rate of interest, p_n = supply price of the nondurable input, and p_d = supply price of the durable input. In the relationship, the user cost of the nondurable input does not, whereas that of the durable input does, involve the rate of interest. The models I have in mind as of particular interest are the horizontal innovation model of Barro and Sala-i-Martin (2004), the creative destruction model of Aghion and Hewitt (1992), and their numerous variants.[1]

For those authors who use the accounting relationship of type (1), the difference in the user cost between the different types of inputs is the natural result of continuous-time model-setting where the nondurable input is used up at an instant whereas the durable input does not depreciate. A strand of reasoning for this, taken so much for granted as seldom to be given an explicit exposition, seems to run as follows. The user cost of an input that

[1] My particular attention to these recent models is due to an essential feature they share: production of commodities by means of commodities, or sequentially connected production. For it is this feature that is particularly vulnerable to the problems associated with an instantaneous nondurable input. See Section III below.

depreciates "radioactively" at the rate of δ is $(\delta + r)p$ where *p* is the supply price of the input and *r* the rate of interest. If an input does not depreciate at all, then $\delta = 0$; therefore, the user cost of the input is rp. If, by contrast, an input depreciates fully in a negligible interval of time (thus, depreciation becomes of the "sudden death" type), then $\delta = 1$ and the rate of interest applicable to the case also becomes negligible—that is, zero in the limit; therefore, the user cost of the input is simply *p*. Thus, regarding the latter type of input, one observes Acemoglu (2009, p. 435) stating that "Since machines depreciate [fully] after use [at time *t*], $p^x(v,t)$ [the price of a machine of variety *v* at time *t*] can also be interpreted as a rental price or the user cost of this machine," with time *t* here obtained by "making the time units as small as possible, that is, by going to continuous time" (p. 47).

An attentive reader, however, may have noted that the above reasoning involves asymmetry in two ways (apart from the fact that depreciation of the "radioactive" type applies to a durable input whilst depreciation of the "sudden death" type applies to a nondurable input). First, there is asymmetry in the treatment of the rate of interest. For both types of inputs, *r* is defined in reference to a certain positive interval of time such that one works, for example as in Hicks, with the "weekly" rate of interest. Now, for one type of input, this "weekly" rate of interest is "shrunk" in proportion with the length of time during which the input is in operation; whilst, for the other type of input, the duration of input operation leaves untouched the length of period that the rate of interest is applied: the rate of interest relevant to the user cost continues to refer to a "week" even if the input is in operation for an infinite run of time. Second, being the other side of the same coin, there is asymmetry in the length of time over which the user cost is defined. The user cost for one type of input is defined for the "shrunk" period of time (an instant), with both the rate of depreciation and the rate of interest (the latter not appearing because it takes the value of zero) applying to an instant, whereas that for the other is defined for the non-shrunk period of time (a "week"), with both the rate of depreciation and the rate of interest (the former not appearing because it takes the value of zero) referring to a "week." A natural question will then arise: what length of time the output (and also the quantity of labor) in (1) refer to?

One would thus have a sufficient (if preliminary) reason to be suspicious of the reasoning in question and to call for a closer look at the derivation of the user cost of an input at a more fundamental level. The user cost of an input is derived from the following familiar formula, in the setting of continuous discounting:

(2) $$PV(\zeta) = \int_0^{\zeta} \pi_d e^{-r\nu} d\nu ,$$

where ζ = the length of time in which the input participates in production (more on this in the following section); π_d = a flow of net revenue arising from one unit of the input over one unit of production period (a "week"); and r = the average level of the ("weekly") rate of interest over ζ.[2] One has $\zeta \to \infty$ for the ever-lasting durable input. Then, the arbitrage condition that, with constant prices in balanced growth equilibrium, the present value obtained in the way of (2) equals the supply price of the input yields the user cost of the input concerned: rp_d. Meanwhile, the user cost of an instantaneous nondurable input is understood in the way of the Hicksian argument, with this argument having never been given any explicit *rigorous* treatment through formula such as (2).

The first objective of the present paper is thus to provide a rigorous way of justifying the Hicksian argument. This objective is subsidiary, however. The main objective is to argue that reasoning in the way of the Hicksian argument on the one hand, applied to an instantaneous nondurable input, and reasoning in the way of discounting formula (2) on the other, applied to an (ever-lasting) durable input, cannot stand side by side, *though each capable of logical validity on its own*. It is suspected that there is no way whatsoever to establish the internal consistency of the accounting relationship of type (1). The implication is that the accounting relationship of type (1) should be revised.

Section I scrutinizes and rigorously justifies the Hicksian argument on the basis of set theory (ZFC). The Hicksian argument justified, there are important implications,

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[2] The continuous-time setting allows the level of net revenue and the rate of interest to vary over time (for example, $\pi_d(v) = e^{-\delta v}\pi_d(0)$ in the case of radioactive depreciation at the rate of δ). Throughout the present paper, however, the simplifying assumption that they are constant over time shall be adopted, with no loss of generality.

hitherto unnoticed, for a (indeed, any) *durable* input used alongside an instantaneous nondurable input. To show that these implications point to some serious logical difficulties is the task of the next two sections. This will lead the reader to Georg Cantor's mathematics of transfinite numbers (Section II) and Zeno's paradox (Section III). The last section concludes.

I. The instantaneous life of a nondurable input: production in "no" time

There is a matter that makes no difference whatsoever in usual cases but which will prove critical in the case of our current concern. That is the way of understanding the upper limit of integration ζ in formula (2). We must distinguish between two such ways.

Definition 1a (the τ-way). The upper limit of integration ζ in formula (2) is defined as the physical time-duration (τ) between the beginning (t_0) and the end ($\bar{t}$) of the lifetime of the input concerned, both t_0 and $\bar{t}$ standing for time coordinates in the Newtonian absolute space-time; hence, simply,

(3) $\zeta = \tau \equiv \bar{t} - t_0$.

Call this definition the "τ-way" (of understanding ζ).

Definition 1b (the *T*-way). The upper limit of integration in formula (2) is defined as the number of "periods of production" (*T*) that the input concerned goes through during its lifetime:

(4) $\zeta = T \equiv \dfrac{\bar{t} - t_0}{u}, \quad u > 0$

where *u* = the physical duration of the unit period of production. Call this definition the "*T*-way" (of understanding ζ).

The unit period of production is the minimum length of time over which a round of production is completed so that a batch of identifiable output is produced and a corresponding flow of revenue generated (even if recorded for pure accounting purpose). The unit period of production is also the *effective* normalization unit of time for a model at hand, because each period of production is associated, by definition, with a *new* measurement of a flow of revenue. In order to be consistent in the matter of measurement, then, all the other *flow* variables in the economy—including the rate of interest, whose denominator is a flow of interest accruing during a certain period of time—are to be measurement-synchronized with the flow of revenue. Now, when $u > 0$, one can always set u as unity. Therefore, the two ways of understanding the upper limit of integration in (2) are equivalent if the period of production is of positive length.

The setting of discrete time requires the unit period of production to be positive. Then, in either way of understanding ζ in (2), a nondurable input is one with $\tau = T = 1$ and a durable input one with $\tau = T = 2, 3, \cdots, \infty$; indeed, *this* is the established definition of a nondurable and a durable input.[3] The setting of continuous time does not exclude the

[3] The classification of inputs into "nondurable" and "durable" is not according to their physical properties but arbitrary up to the choice of the normalization unit of time: the model maker decides upon the normalization unit of time for her model, and any input that is used up just over *one* normalization unit of time is classified as "nondurable," and any input that is used over a multiple number of normalization units of time is christened "durable." One and the same input, which has an engineering-determined physical lifetime, can be classified as either nondurable or durable depending on the chosen normalization unit of time. This consideration buttresses my argument for the equivalence between the effective normalization unit of time and the unit period of production.

If, in the manner of "point-input, flow-output" production, an input is (physically) used up in one normalization unit of time but (economically) generates revenue over a multiple number of units of time (including the case where an input is used just once and revenue is also generated only once but after the lapse of some units of time in which no revenue ensues—a case where π is not a constant function of time), this input must be treated as *durable*. In the case where no nondurable input is used, the concept of the period positive unit period of production; it is conceivable that output is generated in discrete time whilst, for example, discounting is done continuously. But what concerns us in relation to the Hicksian argument is that the unit period of production is an instant and that the duration of an instant is zero (an instant of time is isomorphic to a point on the real half line and the Lebesgue measure of a point is zero).[4] Then we shall be confronted with the problem that not only cannot the *T*-way be used for the case of $u = 0$ but also is the τ-way incompatible with the Hicksian argument.

In the τ-way, a non-depreciating durable input has no end of its (physical) lifetime: $\tau \to \infty$. Then, formula (2) gives rise to the expected formulation of the user cost for a non-depreciating durable input. But this is not the case with an instantaneous nondurable input. As the lifetime of the input is "shrunk" to an instant, the domain of integration is accordingly "shrunk" to the degenerate interval of time [0, 0] in the limit; that is, $\tau = 0$. The relevant amount of revenue will have been, as it should be, time-scaled to comply with an instant of time; as "input produces output instantaneously" (Hicks, 1973, p. 21), however, the amount of instantaneous revenue is positive. The "weekly" rate of interest is finite.

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of production becomes ambivalent and an arbitrary normalization unit of time plays the role of time-scaling economic variables.

[4] Hicks uses the term an instant of time in the meaning of the limit of an ever-shrinking time interval, in the framework of standard calculus. My construal of the term is slightly different—as a point on the real line; I have chosen this construal for it easily allows one to turn to set theory (ZFC). In the case of an instantaneous nondurable input, construal in terms of a limit is much simpler than the one in terms of the set theory, but the latter is more convenient and rigorous when argument is transferred to a durable input. One can also use the term to mean an "infinitesimal" time interval to get engaged in non-standard calculus (*e.g.*, Robinson, 1996; an infinitesimal number is a non-standard number, which is not zero but less than any positive number). It turns out (as it should) that all these alternative understandings lead to the same results. Relevant reasoning based on standard analysis (*i.e.*, in terms of limit) is positioned in various footnotes below.

Proposition 1. Formula (2), applied in the τ-way, fails to generate the user cost of an instantaneous nondurable input.[5]

Proof. For an instantaneous nondurable input, the lifetime of which is an instant, one should have $\tau = 0$. Then, in accordance with formula (2),

$$(5) \qquad PV(\tau)\big|_{\tau=0} = \int_0^0 \pi_n e^{-r\nu} d\nu = 0 .$$

This is contradictory to the fact that the input generates, though instantaneously, a positive amount of revenue and thus that the revenue, discounted at a finite rate of interest, must also be positive.[6] □

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[5] In contrast to this argument of mine, Hicks (1973; "Note to Chapter II") shows how the rate of interest, which is applied to the current-period net output in discrete-time capital-value accounting, disappears in continuous time. His mathematics is immaculate—but only up to the point where one is ready, against logic as will be shown below, to accept his use of the "instantaneous" rate of interest in the continuous discount factor. The reader is also to keep it in mind that Hicks is dealing throughout his book with a productive process (or the economy) where a durable input is the sole material input in operation, not with a productive process (or the economy) where a durable input is in operation along with an instantaneous nondurable input—the case which is represented by the accounting relationship (1) and which is subject, I argue, to the various logical problems that are to be unearthed in the present paper. Moreover, Hicks's case of a singleton of net output (p. 21) is one where a durable-input-using process is "truncated" at the first round of production, not outright a productive process where a nondurable input alone is in operation. An interpretation of the Hicksian argument in the way of (10) or (10') below (which also follows the τ-way) seemingly explains the non-interest-bearing of an instantaneous nondurable input. Criticism of this interpretation shall be on the waiting-line until some necessary prerequisite ideas are discussed.

[6] With a positive and finite supply price ($p > 0$), the user cost of a nondurable input would have to be infinite.

Definition 1b is inapplicable, of course, unless on the assumption that $u > 0$. The state of affairs one is now faced with is where $u = 0$ and, at the same time, $\bar{t} - t_0 = 0$: (4) would be dividing zero by zero. It is obvious that one cannot use formula (4) to obtain the number of periods of production covered by the lifetime of a nondurable input that is used up at an instant. However, one can provide, *along the T-way* suitably modified, a rigorous explanation of the user cost of an instantaneous nondurable input appearing in the accounting relationship (1).

The number of periods of production a nondurable input goes through during its lifetime must be, by definition, *unity*: a nondurable input is an input that is "used only *once* in some productive process" (Silberberg and Suen, 2001, p. 387, emphasis added). To obtain that number in a rigorous manner, which will apply whether the unit period of production is a positive interval or an instant, one turns, I propose, to set theory (ZFC).

Definition 2 (the number of periods of production). The number of periods of production covered by the lifetime of an input is defined as the *cardinality* $|S|$ of the partition set *S* of that period, each partition cell having the length of the unit period of production.

Then, when the unit period of production for a productive process is an instant, the partition set *S* of the period during which an instantaneous nondurable input is operated is[7]

(6) $S \equiv \{\mathcal{I}_i = [t_i, t_i] \subseteq S\}$ such that $\forall i, j \in \{0\} \cup \mathbb{R}_+ \; (i \neq j)$, $\mathcal{I}_i \cap \mathcal{I}_j = \varnothing$ and $\cup_i \mathcal{I}_i = [t_0, \bar{t}]$.

Here, $\mathcal{I}_i$, representing the unit period of production, is a degenerate interval, that is, a point, and *S* is in fact the set of the real numbers in interval $[t_0, \bar{t}]$, each real number constituting a partition cell for the partition set *S*. It is an easy step from here to *Proposition*

[7] The reader can easily construct *S* for the case of a positive period of production.

2.

Proposition 2. The number of production periods contained in the lifetime of an instantaneous nondurable input is unity. [8]

Proof. Because an instantaneous nondurable input lasts for a single instant (t_0), S is the singleton consisting of that single instant, and its cardinality is unity:

(7) $S = \{\mathcal{I}_0\}$; therefore, $|S| = 1$. □

This is a logical explanation of the *single* occurrence of production and revenue-generation—and its accompanying *single* event of discounting (which will actually turn out to have no effect)—which takes place in a period of *zero* length when the normalization unit of time is also a period of *zero* length.

This alone, however, does not complete the story of discounting where the unit period of production is an instant. The instantaneous unit period of production means that the effective normalization unit of time for a model at hand is also an instant. All the *flow* variables in the model are to be measured, rather oxymoronically, in reference to an instant. The flow of revenue and the rate of interest are no exceptions. One should be talking of the "instantaneous flow" of revenue and the "instantaneous rate" of interest: the former is the measure of revenue ensuing at an instant, and the latter is the rate of interest that is defined in reference to an instant (*i.e.*, *not* to a positive, however small, interval of time).[9] Now,

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[8] The same result can be obtained by considering an instant to be the limit of an ever-shrinking interval. Denote the duration of time from moment t_0 to moment t by $\nu \equiv t - t_0$; then, an instant as the limit of an ever-shrinking interval is $d\nu \equiv \lim_{t \to t_0} (t - t_0)$. Since both the lifetime of the nondurable input and the unit period of production are an instant, one gets from formula (4): $T = \frac{d\nu}{d\nu} = \frac{\lim_{t \to t_0}(t - t_0)}{\lim_{t \to t_0}(t - t_0)} = 1$ (by L'Hôpital's rule).

[9] Thus, Hicks (1973) is careful enough to refer, when arguing in discrete time, to "the rate

recall that the continuous discount factor in formula (2)

$$(8) \qquad e^{-r\nu} \equiv \lim_{H \to \infty} \left(1 + \frac{r}{H}\right)^{-H\nu}$$

is defined on the basis of the procedure by which the unit period, *to which r refers*, is divided into H smaller intervals, with H approaching infinity: r is shrunk in proportion with the duration of time to which it is applied each round, and ν stands for the total duration of time to which (or the total number of rounds in which) r is applied. The continuous discount factor requires for its definition that the normalization unit of time, to which the rate of interest refers, should be a *positive* interval—an interval which is subsequently to be divided into an infinite number of smaller (infinitesimal) intervals. But the fact is that we are now in the "instantaneous" world where all "flow" variables, including the flow of revenue and the rate of interest, are measured in reference to an instant—and an instant is, by definition, *indivisible*. Thus, we have the following *Proposition 3*.

Proposition 3. When the unit period of production is an instant (and, thus, when an instantaneous nondurable input is used), one cannot use the continuous discount factor ($e^{-r\nu}$).[10]

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of interest for one week" (p. 20) and to denote it by r throughout his book, whilst referring to the "*instantaneous* rate of interest" (p. 24, emphasis in original) and denoting it by a new variable, ρ, when working in continuous time.

[10] Hicks's carefulness, mentioned in the footnote just above, is thus spurious. Should he have wished to continue to use the continuous discount factor, he should have continued to work with the "weekly" rate of interest and, as a result, work in discrete time. His real intention seems to be that he wishes to deal with the instantaneous period of production and, in accordance, with the instantaneous rate of interest; then, he should have abandoned the continuous discount factor. By contrast, recent theories always use the "weekly" rate of interest when working with the continuous discount factor and thus are safe from this kind of spuriousness; however, this safety may rather be, one cannot help being suspicious, an

Proof. Presented above. □

The indivisibility of the unit period of production suggests, thus, that one should use the (*instant-wise*) *discrete* discount factor and obtain the present value of revenues as the sum of a sequence of discrete terms of discounted revenues, not as an integral of continuous flows of discounted revenues.[11] The number of terms in the series of discounted revenues is the number of production periods that an input goes through during its lifetime, and it is calculated as the cardinality of *S*. Thus, the general formula for obtaining the user cost would be:[12]

(9) $$PV(|S|) \equiv \sum_{s=0}^{|S|-1} \pi(1+\rho)^{-s} .$$

Note that we have used ρ, following Hicks (1973, p. 24), to denote the "instantaneous rate of interest" that is defined in reference to an instant (by contrast, *r* in the previous formulations is the rate of interest whose reference period is a positive interval of time such as a "week") and π has been appropriately time-scaled to be an "instantaneous flow" of revenue.

We have thus finally reached a position to provide a rigorous justification of the Hicksian argument for the user cost of an instantaneous nondurable input.

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incidental result of complete inattention, in the very antithesis of Hicks's attitude.

[11] Thus there is an irony here: whilst the use of a positive time interval as the unit period of production (thus, measuring time in discrete terms) allows continuous discounting, the use of an instant as the unit period of production (which points to the measurement of time in continuous terms) would lead to "(instant-wise) discrete discounting."

[12] This formula is similar to that of discrete discounting where the unit period of production is positive and net revenue is realized at the *beginning* of the period (just as in Hicks's "week," where the clearing houses are open only on the Mondays). Note that the instantaneous period of production annuls the distinction between the beginning and the end of a period.

Proposition 4 (the Hicksian argument). The user cost of an instantaneous nondurable input is as in formula (2).

Proof. $|S| = 1$ for an instantaneous nondurable input; therefore, using formula (9),

$$PV(|S|)\Big|_{|S|=1} = \sum_{s=0}^{0} \pi_n (1+\rho)^{-s} = \pi_n (1+\rho)^{-0} = \pi_n . \quad \square \tag{10}$$

This is, I propose, the logically consistent interpretation of the Hicksian argument regarding the instantaneous nondurable input.

Some clarification, which will prove critical soon, is in order regarding the "instantaneous flow" of revenue and the "instantaneous rate" of interest appearing in (9) and (10). One may be tempted to construe the instantaneous flow of revenue as the "average" revenue ($\overline{\pi}_{t_0}$) over an instant t_0, in the same way as one does with a positive period of production; that is, as $\overline{\pi}_{t_0} \equiv \lim_{t \to t_0} (t - t_0)^{-1} \left(\int_{t_0}^{t} \pi(\nu) d\nu \right)$. In this case, L'Hôpital's rule gives $\overline{\pi}_{t_0} = \pi(t_0)$; π and π_n in (9) and (10) represent revenue generated at each instant t_0. Though mathematically spotless and yielding the expected result, this does not concur with the *concept* of the instantaneous period of production. An instantaneous nondurable input generates revenue *once* and for all, *at* an instant (or *at* the limit of the ever-shrinking interval of time), so that the revenue from the input cannot but be represented at once by a singleton $\pi(t_0)$ referring to that single instant (or limit). It is not the case, by the very definition of the instantaneous period of production, that there first exists $\pi(\nu)$ which is defined *ex ante* over a nonnegative time interval $[t, t_0]$ ($t \geq t_0$) and is ready to give rise to the level of revenue which matches the production event happening at an instant t_0 or at a limit $\lim_{t \to t_0} t$. Nor is it the case that we are considering an input that is physically capable of operation for longer than an instant (so that it is capable of generating revenue over a positive interval of time) but runs out of its economic effectiveness beyond an instant: such an input is *ex ante* (*i.e.*, physically) a durable input and the economic effectiveness of an input is to be determined endogenously.[13]

More caution is required in the case of the "instantaneous" rate of interest. It has nothing to do with a rate of interest (r) that refers to a positive period of time. It is not the same, in concept or in size, as the "average" ($\overline{r}_{t_0}$) of r over an instant t_0, because $\overline{r}_{t_0} \equiv \lim_{t \to t_0}(t - t_0)^{-1}\left(\int_{t_0}^{t} r(\nu)d\nu\right) = r(t_0)$; that is, $\overline{r}_{t_0}$ is the rate of interest r prevailing at instant t_0, *referring to a given positive period*. In contrast, the instantaneous rate of interest (ρ) refers outright to an instant.[14]

Now, in contrast with the interpretation of the Hicksian argument expressed by (10), which follows the T-way, some may suggest the following interpretation, which is along the τ-way (but is different from (5), which has been shown to lead to a nonsensical result):

(11) $$PV(\tau)\big|_{\tau \to 0} = \lim_{\tau \to 0} \pi_n(v)e^{-r\tau} = \pi_n(0).$$

Revenue is a continuous function of time so that it is observed and measured at each instant of time t: $\pi_n = \pi_n(\nu)$, $\nu = t - t_0 \geq 0$. With t approaching t_0, π_n approaches a singleton $\pi_n(0)$, which is the revenue generated at instant t_0 as soon as the input is put into the productive process and immediately used up. The corresponding discount factor is $\lim_{\tau \to 0} e^{-r\tau}$, which is unity (that is, no discount).

This interpretation suffers from several difficulties. The first of them is easily brought to light in reference to the clarification I have just made. The interpretation presupposes the existence of function $\pi_n = \pi_n(\nu)$ which defines revenue over a time interval $[t, t_0]$ with $t \geq t_0$; however, the concept of the instantaneous nondurable input precludes the existence of such a function and instead can allow only a singleton $\pi_n(0)$.

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[13] This is Hicks's case of a durable-input-using process being "truncated" at the very first "week," shrunk or not, of its operation.

[14] Nor is ρ the same as the "instantaneous" rate of change of $e^{r(t)t}$ at instant t_0, which is $r(t_0)$ again.

A counter-argument to this criticism may be put forward immediately: the Hicksian argument is indeed describing what is happening *at* an instant so that it should be interpreted as

(11’) $$PV(\tau)\big|_{\tau=0} = \pi_n(0)e^{-r.0} = \pi_n(0)\,.$$

But nor can (11’) escape from difficulties, which are at the same time the remaining difficulties facing (11). Recall that the use of the continuous discount factor implies that *r* refers to a period of time of a positive length. Now, with the unit period of production and hence the effective normalization unit of time set to be an instant, the length of the period which *r* refers to is *arbitrary* in relation to the period of production. The rate of interest may not be entirely arbitrary, of course, in a complete economic model: for example, as Acemoglu (2009, p. 436) states explicitly, “the interest rate … is determined from the prices of (zero net supply) Arrow securities that households can trade to transfer consumption across dates,” with such a household decision being dealt with, whether explicitly or implicitly, in one part of a model at hand. What matters for our immediate concern is, however, that the rate of interest defined such way is arbitrary *in relation to the (instantaneous) period of production* which is adopted in another part of one and the same model. (11) and (11’) are based on the procedure of taking a rate of interest defined in reference to a positive interval of time and then contracting it in accordance with the shortening to an instant of the relevant period. But such a procedure commits an error of inconsistency, that of applying different normalization units of time in one and the same formula: an instant to the flow of revenue on the one hand, and a positive interval of time to the rate of interest on the other.

This inconsistency in measurement leads to a further difficulty, which will prove fatal to interpretations (11) or (11’). This difficulty is encapsulated in *Propositions 5a* and *5b*.

Proposition 5a. An explanation of the Hicksian argument in terms of (11) or (11’) leads to a case of logical nonsense.

Proof. Suppose there is a (durable) input which generates revenue by the amount of

$\overline{\pi}$ every instant but continuously for *longer* than an instant, say for $\tau \in (0,1]$, with $\overline{\pi}$ being the same amount that the instantaneous nondurable input generates for a single instant. The present value of the revenues generated by this input is obtained as:

(12) $$PV(\tau) = \int_0^\tau \overline{\pi} e^{-r\nu} d\nu = \overline{\pi} r^{-1}(1 - e^{-r\tau}) \; < \; \overline{\pi}, \;\; \forall r > 0, \;\; \tau \in (0,1].$$

Suppose further that (11) or (11') is interpreted as describing what is happening at an instant. Then, the present value of flows of revenue which are generated for *longer* than an instant is *smaller* than that of the revenue which is generated at a single instant, even if the amount of revenue which is generated at each instant is the same. □

Fault must be found in the interpretation of (11) and (11'). On the one hand, a weight of *zero* (in the limit or outright) is given to the rate of interest: $r \lim_{\tau \to 0} \tau$ or $r \cdot 0$. This weighting is in order to reflect the physical duration of time involved with the instantaneity of the nondurable input. Of great importance is that this weighting is required precisely because the measurement unit of time applied to the rate of interest is a *positive interval*. On the other, a weight of *unity* is given to the instantaneous flow of revenue: $\pi(0) \cdot 1$. This weighting must be to reflect the fact that revenue-generation by the nondurable input has indeed taken place and that for once (*i.e.*, the number of period of production is one). As what is being weighted is the instantaneous flow of revenue $\pi(0)$, the weighting is consistent with another fact that the measurement unit of time applied to revenue is an *instant*. One is thus standing perilously on thin ice, which is bound to give way due to the imbalance of weight—the thin ice of inconsistency in relation to the measurement unit of time in one and the same model (or, rather, one and the same formula).

Along the τ*-way*, giving the weight of unity to the revenue generated at an instant is equivalent to expanding the physical duration of revenue-generation to the unit period $[0, 1]$ from the degenerate interval $[0, 0]$. Whether this expansion, if understood as a re-scaling of time, is a logically valid way of doing so is one problem.[15] Still another problem,

[15] This is, so to speak, granting existence (the weight of unity) to nothing (an instant of

which has been revealed in result (12), is reported as *Proposition 5b*.

Proposition 5b. If one interprets the Hicksian argument in terms of (11) or (11'), one is violating the setting of continuous discounting.

Proof. Interpretation (11') (and *mutatis mutandis* (11)) can be construed as

(11") $$PV(\tau)\big|_{\tau=0} = \pi_n(0)e^{-r.0} = \int_0^1 \pi_n(0)\, dv$$ (see also Figure 1).

The right hand side of the last equality is an expression where *no* discounting is done over the time interval [0, 1]. □

<Figure around here>

That is, whilst the period during which an instantaneous nondurable input is in operation is re-scaled from an instant, whose length is zero, to a positive interval whose length is unity, no part of the revenue generated over that re-scaled period of time is discounted.[16]

If one wishes to achieve consistency in this matter and, at the same time, insists on using the continuous discount factor, one should have, instead of (11) or (11') (and putting aside the first problem regarding the singleton of revenue),

(13) $$PV(\tau)\big|_{\tau\to 0} = \lim_{\tau\to 0}\left(\pi_n\tau\right)e^{-r\tau}.$$

π_n now stands for a flow of revenue generated over the same *positive* interval of time to which r refers; multiplier τ represents the physical duration of the lifetime of the input. Weighted by τ (not unity), π_n is time-scaled *pari passu* with r (which is also weighted

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time, whose Lebesgue measure is zero). Detailed and critical discussion of this way of rescaling time is found in the working paper version of the present paper.

[16] Note that it is the use of continuous discount factor which forces the expansion of the period—hence, further support to the non-applicability of the continuous discount factor in our instantaneous world.

by τ). One shall easily see, however, that (13) is nothing but an alternative expression of (5). We are right back to the starting point: consistency achieved, result nonsensical.

Thus, as *Propositions 1, 5a* and *5b* together demonstrate, decisive evidence is that the τ-way—discounting in reference to the physical duration of lifetime of an input—cannot consistently explain the user cost of an instantaneous nondurable input. One zero alone can be nothing but zero, as in (3). But two zeroes, one "divided" by another, as in (4), may produce unity. *Proposition 4* is the evidence. The result of the proposition is obtained by the *T*-way—discounting in reference to the number of periods of production covered by the lifetime of an input—and expresses, in a logically and conceptually consistent way, the single occurrence of production, of revenue-generation and of the corresponding discounting with regard to a nondurable input which is used up once and for all, in a time interval of zero length.

Logic, saving an instantaneous nondurable input, however, does not let pass without a toll. The toll falls on the other member of the production partnership—a *durable* input.

II. The "eternal" life of *any* durable input: counting the uncountably infinite

If a nondurable input is an input that is exhausted at an instant, a durable input is any input that lasts for longer than an instant—that is, any input that stays in production for a *positive* interval of time, no matter how short.

Lemma (Georg Cantor's $\aleph_1$). The cardinality of any non-degenerative interval in the real numbers $\mathbb{R}$ is $\aleph_1$ (aleph-one).

Proposition 6. The number of production periods covered during the lifetime of any durable input (an input that lasts for longer than an instant) is $\aleph_1$.

Proof. The unit period of production is an instant, which is isomorphic to a point on the real line. The lifetime of any durable input is isomorphic to a positive interval in $\mathbb{R}$.

Then, *Definition 2* and *Lemma* together give the result. □

This result leads us to the following *Proposition 7*:

Proposition 7. Formula (9), which successfully applies to an instantaneous nondurable input, fails to hold for a durable input.

Proof. Applying formula (9) for any durable input, whether of finite or infinite duration in calendar time, one will have, using *Proposition 6*,

(15) $$PV(|S|)\Big|_{|S|=\aleph_1} = \sum_{s=0}^{\aleph_1 - 1} \pi_d (1+\rho)^{-s} .$$

This "series" does not converge unless at most countably many terms in the sequence are nonzero (see Bonar and Khoury, 2006, p. 94). But this is not the case with the durable input, because a nondurable input is used up at every instant and, therefore, in accordance with the definition of the period of production, a positive amount of output (and thus revenue) is generated at every instant during the lifetime of the durable input. □

Formula (15) is expressed as the sum of a sequence of discrete terms. Now, a sequence of discrete terms, in its ordinary notion, is indexed by the *natural numbers*: there is a one-to-one correspondence between its "*i*th" term and a natural number *i*. A sequence involves listing its terms, and this listing amounts, in the ordinary notion of a sequence, to *counting* the terms following the sequence of the natural numbers. If a sequence is *infinite* (in the sense in which we use ∞ for the number of terms in the sequence), the number of its terms is the same as the number of all the natural numbers, $\aleph_0$ (aleph-null). In this case, the following familiar formulation applies:[17]

(16) $$\sum_{s=0}^{\infty} \pi_d (1+\rho)^{-s} = \pi_d (1+\rho^{-1}) .$$

By contrast, the number of terms in "series" (15) is $\aleph_1$, which is "larger" than

[17] One shall note that this gives a result different from that of continuous discounting (2).

$\aleph_0$.[18] "Series" (15) proposes to sum *more* than the *infinite* (∞) number of terms that the usual infinite series (16) refers to. Looking at the opposite side of the same coin, series (16) leaves *uncounted* some terms that are supposed to be included in series (15). Indeed, Cantor tells us, there are in (15) infinitely more uncounted terms than counted ones. Formula (15) refers to a sequence indexed by the real numbers in a positive interval—to be strict, a *net* (or a Moore-Smith sequence) on a positive real interval. The sum of a sequence indexed by an uncountable set converges only if at most countably many terms in the sequence are nonzero (see, *e.g.*, Bonar and Khoury, 2006, p. 94). But series (15) has uncountably many nonzero terms—it does not converge to a finite number. (15) squarely fails, in the setting of the instantaneous period of production, as a formula for obtaining the present value for a durable input.[19]

Putting aside this difficulty for a while, let us assume that (16) is a valid formula for a durable input, only to see that the assumption engenders another difficulty. A nondurable input being defined in reference to an instant, any input that lasts for longer than an instant is classified as durable. But durable inputs are not all the same, as they may have different physical lifetimes: one lasting for a second, another for a day, a third for a millennium. But with the instantaneous period of production, the number of periods of production in which these different durable inputs (indeed, *any* inputs that last for a *positive*

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[18] Cantor's "continuum hypothesis" with the axiom of choice says that $\aleph_1 = 2^{\aleph_0}$.

[19] Consider the matter in terms of the ever-shrinking period of production. For a durable input, one should calculate $T = \dfrac{\overline{t} - t_0}{d\nu}$ with $\overline{t} > t_0$. Even though a "formal calculation" (in the sense used in mathematical logic) gives the result of ∞, more rigorous branches of mathematics conclude either that the expression is meaningless, or that it gives rise, depending on the values of $\overline{t} - t_0$, to different "infinite" values which are not equivalent between them. (Should different values of $\overline{t} - t_0$ be associated with the same "infinite" values, one would be faced with the problem reported as *Proposition 8* below.)

period of time) are in operation is, if (16) applies, uniformly "infinite" (∞).[20] Then, inconsistency ensues.

Proposition 8. Should one use formula (16) for a durable input with the instantaneous period of production, logical nonsense would result.

Proof. Suppose the flow of revenue generated in each instant is the same across durable inputs of *different* lifetimes. Should formula (16) apply, the number of production periods covered by these lifetimes would be uniformly ∞; therefore, the inputs would generate the same discounted present value regardless of their different lifetimes and despite the fact that the same amount of output is produced across the inputs at each instant. □

Under formula (16), different lifetimes of the durable inputs (which are the *physical* criterion by which to distinguish between them) are of no relevance at all in identifying the number of periods of production they go through during their lifetimes and, therefore, the total amount (discounted or not) of revenue they respectively give rise to (which are the sole *economic* criterion by which to distinguish between them): an input that lasts for a nano-second and another that lasts for a millennium would, even if it is supposed that the amount of revenue generated at each instant is the same for both inputs, generate the same amount of revenue, whether undiscounted or discounted, during their lifetimes.[21]

This upsetting result comes from understanding the domain of discounting in the *T*-

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[20] If a durable input lasted for "$n(\in \mathbb{N}, \geq 2)$ instants," the number of period of production for the input would be n. This statement, of course, holds only if we could *count* the number of instants and accept that there was an input that lasted for a multiple (finite) number of instants but never for a positive interval of time.

[21] The same difficulty is exposed when one considers the matter in terms of the limit and applies a "formal calculation" (see the previous footnote): for *any* $\bar{t} > t_0$, one obtains uniformly $T = \infty$.

way—in terms of the number of production periods that an input goes through during its lifetime spanning the time interval $[t_0, \bar{t}]$. One may thus counter-argue, on the ground that $\pi_d(1+\rho)^{-s}$ is a continuous function of $s \in [t_0, \bar{t}]$, defined on a real interval. This would allow one, it may be suggested, directly to take the time interval as the domain of discounting—that is, to follow the τ-way—and to resort to *integration* over that domain. (This is to work in the setting of continuous discounting whilst maintaining the discrete-discounting form of each term). For the sake of argument, again, let us assume that such reasoning is justified.[22] One would then have for a durable input which lasted for the duration of $\tau = \bar{t} - t_0$,

(17) $$PV(\tau) = \int_0^\tau \pi_d (1+\rho)^{-s} \, ds = \frac{\pi_d}{\ln(1+\rho)} \left[1 - (1+\rho)^{-\tau} \right].$$

For an ever-lasting durable input one would have the present value as $\pi_d \left[\ln(1+\rho) \right]^{-1}$—a result that is not what one wishes rigorously to have to match the accounting relationship (1).

The real difficulty of this proposal would yet to come, however. Integration would backfire—on a *nondurable* input. The duration of lifetime of an instantaneous nondurable input is zero (in the limit or outright); hence, one should have

(18) $$PV(\tau)\big|_{\tau \to 0} = \lim_{\tau \to 0} \int_0^\tau \pi_n (1+\rho)^{-s} \, ds.$$

We would have made a full circle again, landing back to precisely where we started: just as in (5), one would invariably have the *zero* present value of revenue for the instantaneous nondurable input.

Thus, the setting of the instantaneous period of production creates intense tension between a nondurable input and a durable input, despite the fact that they are synchronized with each other to produce output in one and the same productive process. It seems this tension that has made Katzner (1988) resort to some schizophrenic solution in his attempt

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[22] This amounts, however, to making the unwarranted assumption that real series (14*) is convergent.

to tackle a similar problem as that under current scrutiny. He arrives at the general continuous-discounting formula

(19) $$PV(\tau) \equiv \int_0^\tau \pi e^{-rv} dv = \pi r^{-1}(1 - e^{-r\tau}),$$

understanding the upper limit of integration as the physical lifetime of an input concerned—that is, in the τ-way. He then refers to the "conventions" that "[the life of a nondurable input] is expressed in terms of numbers of instants (that is, numbers of infinitesimal lengths of time) and assumed to be unity" whilst "lives of durable inputs are scaled as numbers of periods of finite length such as years and months," so that "[the life of a durable input] can be any positive real number" (p. 386). From this he infers that "the limit of [the life of a durable input] as [it] declines is zero—not the life of a nondurable input" (p. 386).

In this attempt, Katzner is willing to allow for different normalization units of time in one and the same model. He is also blind to the close relationship, and the necessary distinction, between the physical lifetime of an input and the number of production rounds it goes through. This leads to his absurd conclusion that some durable inputs would have a shorter lifetime than a nondurable input. Further, he does not realize that, his statements notwithstanding, he is actually using a common unit of time for all kinds of inputs—a (any) positive interval of time. This is confirmed by his concept of an "instant" of time: "the concept of 'instants of time' hereafter is characterized specifically as an arbitrarily small time interval. The alternative possibility of thinking of an instant as the limit of a sequence of smaller and smaller intervals … no longer is allowed" (p. 380). When he actually comes to an instant of time of his alternative thinking, he makes a (correct) statement that "it would be necessary to set [the life of an input] = 0 for nondurable[s]" (p. 386, fn. 9), that is, $\tau = 0$ in (19), and then arrives at the (mathematically correct but economically embarrassing) conclusion that "their rental values would always be infinite" (p. 387, fn. 11). He is not aware that the derivation of formula (19) is at the outset based on a *positive* period of production (his "arbitrarily small time interval"); one cannot use the formula, as we have shown, if the period of production is an instant (his "limit of a sequence of smaller and smaller intervals”). The correct formula with the instantaneous period of production is (10); however, then, unbearable burden is put on the shoulder of a durable input.

The problem facing a durable input is the oxymoronic difficulty of counting uncountably many production rounds. This has resulted from the instantaneity of the period of production, which leads one inevitably to consider the matter in terms of the real numbers. This association with the real numbers is the source of a further difficulty.

III. The non-economics of instantaneous production: Zeno’s paradox

Economic—indeed, any physical—events, measured in terms of separately complete units, take place either simultaneously or one “next” to another over time. Take the example of horizontal innovation (or creative destruction) models.[23] The economy begins, at time zero, with given stocks of the final good, intermediate goods and designs. Production takes place simultaneously in each of the three sectors to produce designs, the final good and intermediate goods respectively. Thanks to this production, the economy is endowed with the new stocks of the inputs to be used for the “next” production. Too obvious though it may sound, the “next” production cannot take place at the same time as the “previous” production. If it did, this would mean that *everything* happened at the same time. The very thesis of horizontal innovation or creative destruction models is that the increasing variety or improving quality of intermediate goods through innovations is the source of continual economic growth. If everything happened at a single instant, one would be allowed to talk about neither “increasing variety,” nor “improving quality,” nor “innovations.” The models

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[23] Examples directly related to this argument are Barro and Sala-i-Martin (2004, ch. 6) and Aghion and Hewitt (1992), which respectively represent horizontal innovation and creative destruction models. Though Romer (1990), the canonical model of horizontal innovations, imagines that all inputs in the economy are durable, it can be shown that this involves an internal contradiction; the resolution of the contradiction requires that at least one of the inputs be nondurable. Gancia and Zilibotti (2005) and Aghion and Hewitt (2005), albeit blind to these problems, are excellent surveys of horizontal innovation and creative destruction models, respectively.

in question, if to be meaningful, should be based upon a *sequence* of production.

This requirement of sequentiality comes to sharper relief if the horizontal innovation or creative destruction process of production is construed in an alternative (but equivalent) way. The final good is produced by means of intermediate goods, these latter goods having been produced by means of a design, this design having been produced by means of the foregone final good, this good having been produced by means of intermediate goods, and so on *ad infinitum*. For horizontal innovations and creative destruction to be possible, there must be a sequence of "production-events," which requires these events to take place one "next" to another along the time line. Indeed, this should be the case for *any* economic models where commodities are produced by means of *produced* commodities (the usual Solovian one-sector growth model being another best-known, the simplest, example).

Thus, if the models in question are *both* to take (10) (or (1)) for granted *and* to be meaningful, production must take place *both* instantly *and* sequentially along the time line. But "sequential instantaneity" is the antithesis of the very property of the real numbers on the basis of which the structure of time is construed. Any positive interval on the real line is a continuum, a linearly ordered set of more than one element that is *densely ordered*: with the linear ordering $<$, for every real number x and y such that $x < y$, there is always another real number w such that $x < w < y$. One cannot locate an element lying "immediately next" to a given element of the set. One cannot identify the instant coming "immediately next" to the present instant.[24] A *sequence* of production taking place *instant by instant* is

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[24] There will always be another instant "after" the present instant, but this "after" can be by a second, by a nanosecond, by a yoctosecond, by 100 attoseconds (the shortest time interval ever measured), or by any arbitrarily small duration. It is no wonder that the ultimate measurement unit of time in (current) theoretical physics, a Plank time, has a positive length. The Bing Bang marked Time Zero; then, it took one Plank time (about 5.39124×10^{-44} seconds) for the first-ever "next" event after the Big Bang—the separation of gravitation from the other fundamental forces (electromagnetism, the weak interaction and

inconceivable. The instantaneous period of production cannot accommodate "production of commodities by means of commodities" (Sraffa, 1960)—the essential characteristic of horizontal innovations or creative destruction.

The inconceivability of "sequential instantaneity" implies that no sense can be made of the idea of obtaining, in the models in question, the present value of revenues generated from a durable input, a single flow of revenue assumed to accrue every instant. In these models, the sequence of instantaneous revenues is rendered possible by a sequence of nondurable inputs being newly introduced into production every instant, that is, a sequence of new "production events."[25] A sequence of instantaneous production events being inconceivable, there can be no sequence of revenues to be discounted. Production can never proceed beyond a given instant. Zeno of Elea has resurrected, triumphantly.[26]

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the strong interaction)—to take place. Thus, when we have above mentioned "a sequence of production," we have already restricted the meaning of a "sequence" to an ordinary sequence indexed by the natural numbers, excluding a net (or a Moore-Smith sequence) indexed by any set, including an uncountable set.

[25] This type of production is different from "point-input-flow-output" one. This latter type of production is free of the difficulty being referred to, for a sequence of "production events" is not a feature of it: here, simply a one-time application of an input generates a continuous flow of output (possibly over an infinite run of time and probably at a varying rate period by period): consider the case where a one-time holing of a spring produces water (or, better or worse, oil) forever. Indeed, this productive process is one where only one type of input, a durable input, is utilized.

[26] The directly relevant to the current case is the paradox of "dichotomy": In a race with a tortoise, Achilles must first run half of the initial distance between them; however, before he can complete this half, he must run half of that half; before he can complete the quarter, he must run half of the quarter, and so on; there is no first member in this regression; therefore, Achilles cannot even get started (see, *e.g.*, Salmon, 2001). The less well-known paradox of "plurality" (which is in fact more fundamental than all the other well-known paradoxes) has the direct relevance to the argument in Section II: "the units in a collection

IV. Conclusion: grant a more-than-instantaneous life to a nondurable input

Georg Cantor and Zeno of Elea—both would have a say in the economics of discounting in which an input is used up at an instant of time.[27] Cantor would wonder how the economist could count the uncountable and also, by doing so, obtain a finite value. Zeno would be amazed with the story of the economist who caught up with the tortoise by making, he claimed, a sequence of instantaneous motions. Homer—many Homers, classic or contemporary—has nodded in the economics of (instantaneous revenue) discounting.

Homer the economist should awake and replace the setting of the instantaneous period of production with that of a *positive* period of production. Homer the economist then will revise the accounting relationship (1) into

(20) $$Y = wL + (1+r)p_n K_n + rp_d K_d ,$$

with r, K_n, Y and L all defined in reference to a common positive interval of time (and also the unity rate of depreciation for a nondurable input referring to the same period of time); hence, our initial question consistently answered. This is a revision that is required, not by the realism of an economic model, but by the minimal logical consistency that a theoretical model should satisfy.

One important implication of this revision is for the measure of "capital" in such multi-sector models as those which have been the object of particular attention in the

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can have no size at all: else they would have parts and be not units but collections of units. … [O]n the contrary, there cannot be anything that has no size at all; for there cannot be a thing which if it were added or subtracted from something else would not affect the size of that thing" (Owen, 2001, p. 140; see also Salmon, 2001).

[27] In the working paper version of the present paper, the author also considers what Isaac Newton would have thought: the model-world which is to be conceived with the instantaneous period of production is the world in which time exists only in the "present" (Now) and in the infinitely (or, more precisely, transfinitely) remote future (Eternity), and *never in between*.

present paper, where production of different commodities is sequentially connected. For, in these models, it will now be the case that "interest on the net input of the week (since it was paid for on the Monday) is included in the capital at the end of the week" (Hicks, 1973, p. 30). Romer's (1990) "accounting measure of capital," or Barro and Sala-i-Martin's (2004) measure of the "assets" of the economy, or Jones's (2005) measure of "financial assets"—whatever legitimate and useful role they may play in their models—is none of them a correct measure of the totality of the means of production used in the economy, in the sense of the measure in reference to which the rate of interest on the means of production is calculated in a logically (and economically) consistent way. Yet to come are further implications which the revision in the measure of "capital" will have for some important conclusions of these models, and which may prove fatally troublesome.

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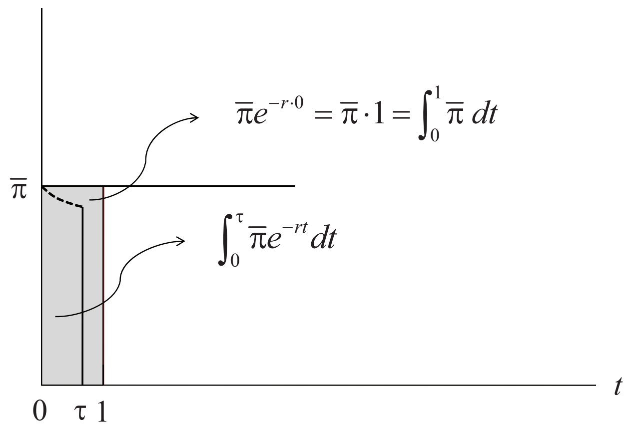

FIGURE. (NO) DISCOUNTING FOR AN INSTANTANEOUS NONDURABLE INPUT